



TITLE:

# Graphic of 3-manifolds and its applications (Low-Dimensional Topology of Tomorrow)

AUTHOR(S):

Kobayashi, Tsuyoshi

---

CITATION:

Kobayashi, Tsuyoshi. Graphic of 3-manifolds and its applications (Low-Dimensional Topology of Tomorrow). 数理解析研究所講究録 2002, 1272: 68-83

ISSUE DATE:

2002-06

URL:

<http://hdl.handle.net/2433/42220>

RIGHT:

# Graphic of 3-manifolds and its applications

奈良女子大学 理学部 小. 林 毅 (Tsuyoshi Kobayashi)

Nara Women's Univ.

## 1. Introduction

It is a classical result of Reidemeister and Singer ([R], [S]) that any two Heegaard splittings of a 3-manifold are stably equivalent. However, it is still unknown that whether there actually exist two Heegaard splittings of the same genus such that more than one stabilizations are required to make them equivalent. In 2001, D. Backman [Ba] introduced a property, called critical, for separating surfaces in 3-manifolds, and give some fundamental results on critical Heegaard surfaces. The author has a feeling that "critical Heegaard surface" could be a breakthrough to the above Stabilization Problem. The purpose of this article is to give a rough sketch of Backman's idea of the proof of one of such results. The author affraid that there are many mistakes caused by misunderstandings of the author's in this arti

Of course the responsibility for such mistakes is due to him. Unfortunately this article contains no new results at all. The author is glad if this article could be a good introduction for Backman's paper.

## 2. Critical surface

Let  $F$  be a separating surface in a 3-manifold  $M$ . Let us call the regions that are separated by  $F$  "red" and "blue".

Definition 2.1 Let  $D, D'$  be compressing disks for  $F$ . We say that  $D$  is equivalent to  $D'$  (denoted by  $D \sim D'$ ) if there exists an ambient isotopy  $\mathcal{P}_t$  of  $M$  such that  $\mathcal{P}_1(F) = F$ , and  $\mathcal{P}_1(D) = D'$ .

(Here  $\mathcal{P}_t$  ( $0 \leq t \leq 1$ ) may not preserve  $F$ , and it is possible that  $\mathcal{P}_1$  exchanges red and blue regions.)

The isotopy invariant disk complex for  $F$ , denoted by  $\Gamma(F)$ , is the 1-complex such that:

- (1) The vertices of  $\Gamma(F)$  correspond to the equivalence classes of compressing disks of  $F$  via " $\sim$ ", and
- (2) Two vertices, say  $v_1, v_2$ , of  $\Gamma(F)$  are connected by an edge if there are compressing disks  $D_1, D_2$  for  $F$  such that:

1.  $D_i$  represents  $v_i$  (i.e.,  $[D_i] = v_i$ ),
2.  $D_1$  and  $D_2$  are contained in different sides of  $F$ ,

and

$$3. |D_1 \cap D_2| \leq 1.$$

Suppose that  $F$  is a Heegaard surface. Then the above conditions 2.3 are equivalent to:

$\{D_1, D_2\}$  is either weakly reducing pair of disks or stabilizing pair of disks.)

We say that a vertex of  $\Gamma(F)$  is isolated if it is not an endpoint of any edge.

Definition 2.2. We say that  $F$  is critical if

$$\Gamma(F) - \{\text{isolated vertices of } \Gamma(F)\}$$

is not connected.

(Roughly speaking,  $F$  is critical if there exists a pair of compressing disks  $D_1, D_2$  for  $F$  such that  $D_1$  is not related to  $D_2$  via pairs of disks giving weak reducibility or stabilization up to the equivalence " $\sim$ ".)

Then, in [Ba], Backman proved the following:

Theorem 1. ([Ba.Theorem 5.1])

Let  $M$  be a 3-manifold with a critical <sup>Heegaard</sup> surface  $F$ . Suppose that there is an incompressible surface  $S$  in  $M$ . Then there exists an incompressible surface  $S'$  which is homeomorphic to  $S$  such that  $F \cap S'$  consists of simple closed curves which are essential in both  $F$  and  $S'$ .

## Theorem 2 ([Ba, Theorem 6.1])

Let  $F, F'$  be mutually non-isotopic strongly irreducible Heegaard surfaces of  $M$ . Suppose that minimal genus common stabilization of  $F$  and  $F'$  is not critical. Then  $M$  contains an incompressible surface.

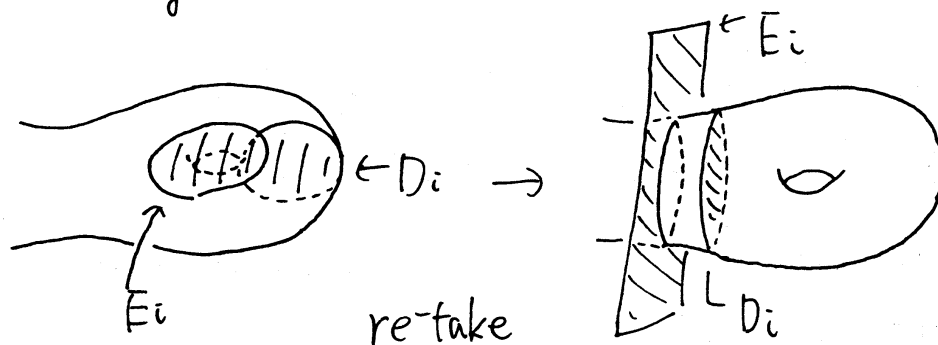
### 3. Sketch of the proof of Theorem 1.

The proof of Theorem 1 is based on the same stuff as Rubinstein-Scharlemann's paper [R-S].

Convention: We denote by  $D_i$  ( $E_i$  resp.) compressing disks of  $F$  contained in red region (blue region resp.).

Stage 1. Constructing the map,  $\Phi: S \times D^2 \rightarrow M$

Since  $F$  is critical, there exist edges  $[D_0] - [E_0], [D_1] - [E_1]$  of  $\Gamma(F)$  which belong different components. By re-taking  $D_i, E_i$ , if necessary, we may suppose that  $D_0 \cap E_0 = \emptyset$ , and  $D_1 \cap E_1 = \emptyset$  (see the figure below).



By using outermost disk argument, it is easy to show that there is a sequence of compressing disks for  $F$ ,  $\{D_{\bar{m}}^{\bar{i}}\}_{\bar{i}=0,1,\dots,m}$ , such that:

$$D_{\bar{m}}^0 = D_0, D_{\bar{m}}^m = D_1, \text{ and } D_{\bar{m}}^{\bar{i}} \cap D_{\bar{m}}^{\bar{i}+1} = \emptyset \quad (\bar{i}=0,1,\dots,m-1)$$

and a sequence of compressing disks for  $F$ ,  $\{E_{\bar{m}}^{\bar{i}}\}_{\bar{i}=0,1,\dots,m}$ , such that:

$$E_{\bar{m}}^0 = E_0, E_{\bar{m}}^m = E_1, \text{ and } E_{\bar{m}}^{\bar{i}} \cap E_{\bar{m}}^{\bar{i}+1} = \emptyset \quad (\bar{i}=0,1,\dots,m-1)$$

Then let  $U_{\bar{i}}$  ( $\bar{i}=0,1,\dots,m$ ) be a neighborhood of  $D_{\bar{m}}^{\bar{i}}$  in  $M$ , and  $V_{\bar{i}}$  ( $\bar{i}=0,1,\dots,m$ ) be a neighborhood of  $E_{\bar{m}}^{\bar{i}}$  with the following properties.

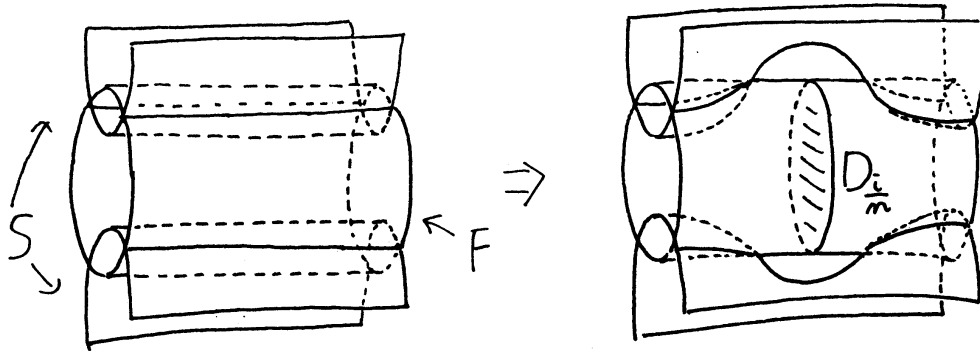
- $U_{\bar{i}} \cap U_{\bar{i}+1} = \emptyset \quad (\bar{i}=0,1,\dots,m-1)$
- $V_{\bar{i}} \cap V_{\bar{i}+1} = \emptyset \quad (\bar{i}=0,1,\dots,m-1)$
- $U_0 \cap V_0 = \emptyset, U_m \cap V_m = \emptyset.$

Let

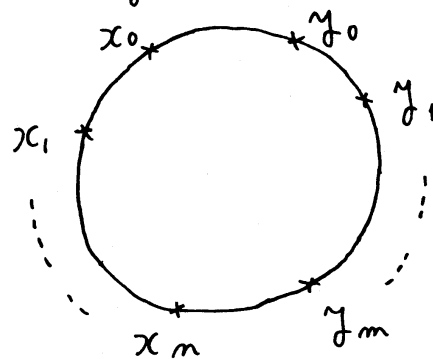
$$\gamma_t^{\bar{i}}(x) : M \times I \rightarrow M \quad (\bar{i}=0,1,\dots,m, x \in M, t \in I=[0,1])$$

be an isotopy with support in  $\text{Int } U_{\bar{i}}$  which pushes  $S \cap D_{\bar{m}}^{\bar{i}}$  out of  $D_{\bar{m}}^{\bar{i}}$ . See the next figure.

Similarly let  $\delta_t^{\bar{i}}$  be an isotopy with support in  $\text{Int } V_{\bar{i}}$  which pushes  $S \cap E_{\bar{m}}^{\bar{i}}$  out of  $E_{\bar{m}}^{\bar{i}}$ .

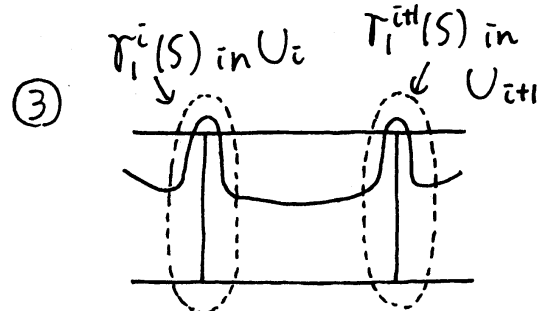
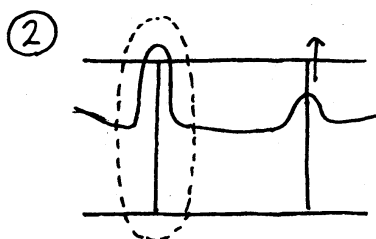
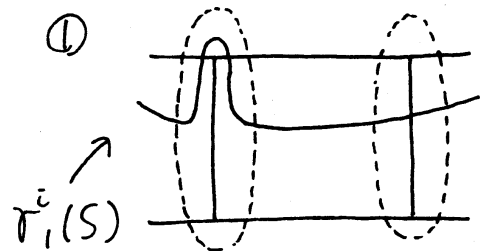
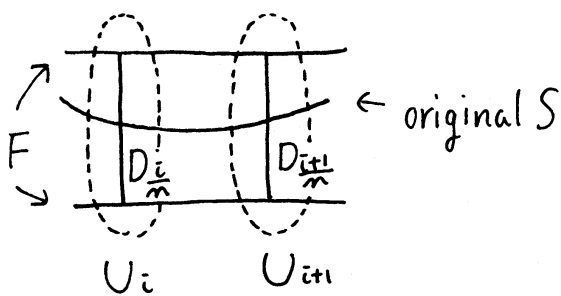


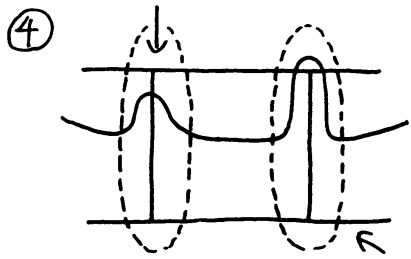
Then we distribute points  $x_0, x_1, \dots, x_m, y_m, \dots, y_1, y_0$  on  $S^1$  as in the following.



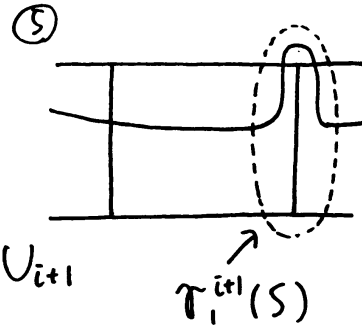
Then we regard  $S^1$  as a parameter space of embeddings of  $S$  in  $M$  as follows:

If we move  $S^1$  as  $\frac{\textcircled{1} \textcircled{2} \textcircled{3} \textcircled{4} \textcircled{5}}{\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \times \quad \quad \times \\ x_i \quad \quad x_{i+1} \end{array}}$ , then





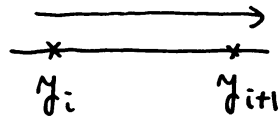
coming back to  
the original position



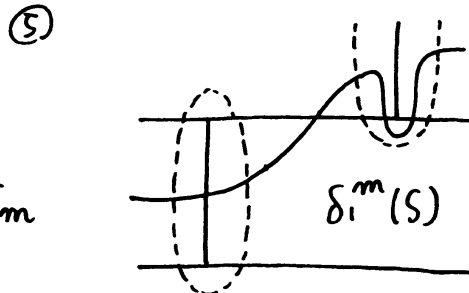
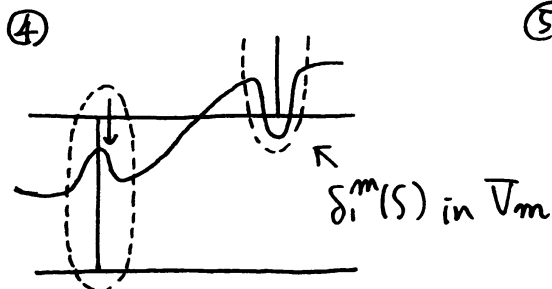
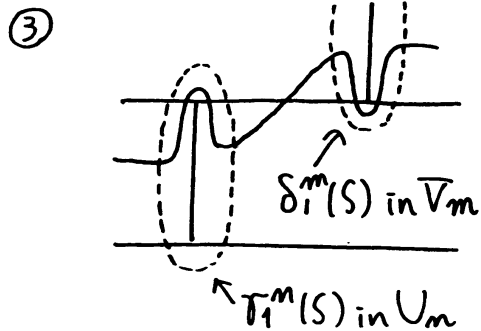
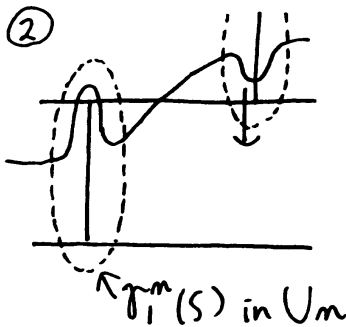
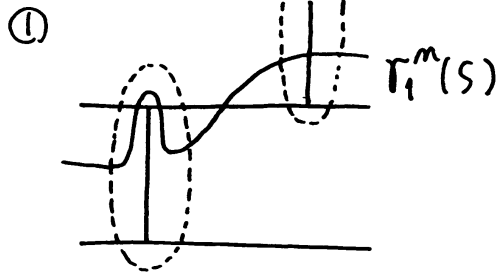
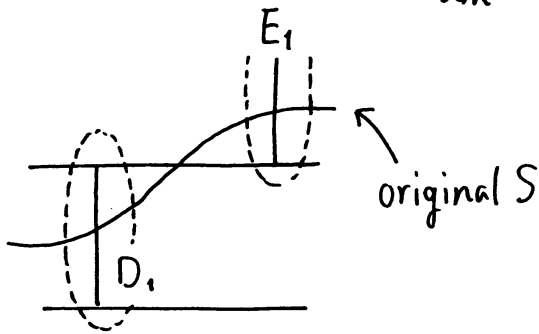
$\gamma_i^{i+1}(S)$  in  $U_{i+1}$

$\gamma_i^{i+1}(S)$

Similarly for



If we move  $S'$  as  $\frac{\textcircled{1} \textcircled{2} \textcircled{3} \textcircled{4} \textcircled{5}}{x_m \quad y_m}$ , then  $S$  changes as



Similar for  $\frac{\textcircled{1} \textcircled{2} \textcircled{3} \textcircled{4} \textcircled{5}}{y_0 \quad x_0}$



Then we extend this parametrization from  $S' (= \partial D^2)$  to  $D^2$  via linear extension, i.e., the center of  $D^2$  is the original  $S$ , and along radial segment, it is deformed linearly to the position corresponding the endpoint in  $S' (= \partial D^2)$ .

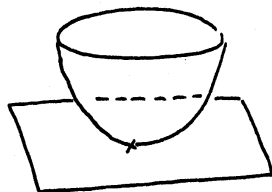
Hence we obtain:

$$\Phi : S \times D^2 \rightarrow M.$$

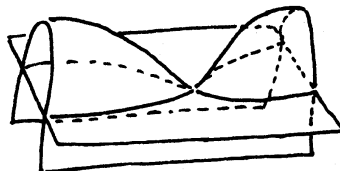
By Cerf theory [Ce], we can perturb  $\Phi$  slightly so that  $D^2$  is stratified into four parts below.

1. Region: Region is a component of the subset of  $D^2$  consisting of points  $x$  such that  $S_x (= \Phi(S \times \{x\}))$  and  $F$  intersect transversely. Region is an open set in  $D^2$ .

2. Edge: Edge is a component of the subset consisting of points  $x$  such that  $S_x$  and  $F$  intersect transversely except for one non-degenerate tangent point. (The tangent point is either a "center" or a "saddle".) Edge is a 1-dimensional subset of  $D^2$ .



center

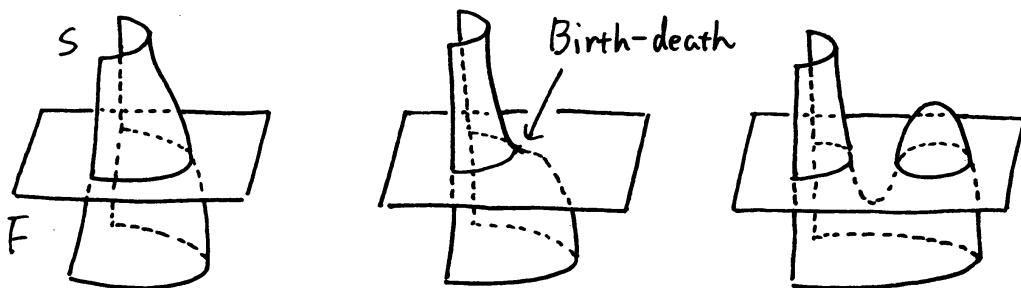


saddle

3. Crossing vertex: Crossing vertex is a component of the subset of  $D^2$  consisting of points  $x$  such that  $S_x$  and  $F$  intersect transversely except for two non-degenerate tangent point. Crossing vertex is a 0-dimensional set (i.e., consists of a point,) and in a neighborhood of a crossing vertex, four edges are coming in.

4. Birth-death vertex: Birth-death vertex is a point  $x$  such that  $S_x$  and  $F$  intersect transversely except for a single degenerate tangent point. In particular, there is a parametrization  $x = (\lambda, \mu)$  of a neighborhood of the point such that:

$$F = \{(x, y, z) \mid z = 0\} \text{ and } S_x = \{(x, y, z) \mid z = x^2 + \lambda + \mu y + y^3\}$$



Birth-death vertex is a 0-dimensional set and in a neighborhood of a birth-death vertex, two edges (one from center tangencies, and the other from saddle tangencies.)

Let  $\Sigma$  be the union of edges and vertices above.  $\Sigma$  is called a graphic.

## Stage 2 Labelling regions

Let  $\mathcal{C}_0, \mathcal{C}_1$  be sets of comp. of  $\Gamma(F)$  such that:

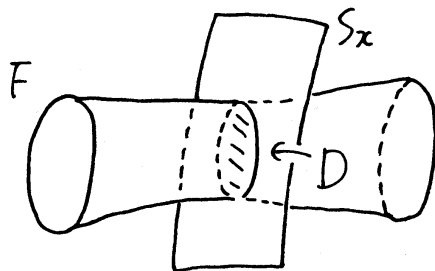
1. Any isolated vertex of  $\Gamma(F)$  is contained in both  $\mathcal{C}_0$  and  $\mathcal{C}_1$ ,
2. Any component of  $\Gamma(F)$  which is not an isolated vertex is an element of exactly one of  $\mathcal{C}_0, \mathcal{C}_1$ ,
3. The component of  $\Gamma(F)$  containing the edge  $[D_0] - [E_0]$  ( $[D_1] - [E_1]$  resp.) is an element of  $\mathcal{C}_0$  ( $\mathcal{C}_1$  resp.).

(For example, let  $\Gamma_0$  be the component of  $\Gamma(F)$ , which contains the edge  $[D_0] - [E_0]$ . Then we may take

$$\mathcal{C}_0 = \Gamma_0 \cup \{\text{isolated vertices of } \Gamma(F)\}, \text{ and}$$

$$\mathcal{C}_1 = \Gamma(F) \setminus \Gamma_0.$$

Let  $R$  be a region. Then we label this region with " $D_i$ " ( $i=0$  or  $1$ ) if there exists a red compressing disk  $D$  for  $F$  such that  $[D]$  is a vertex of an element of  $\mathcal{C}_i$ , and  $D \subset S_x$  ( $x \in R$ ). See figure below.

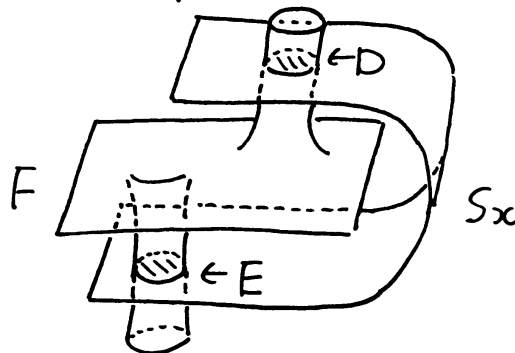


Similarly we label the region with " $E_i$ " if there exists a blue compressing disk  $E$  for  $F$  such that  $[E]$  is a vertex of an element of  $\mathcal{C}_i$ , and  $E \subset S_x$ .

Remark By this definition, we see that if  $\bigcup_{x \in R}$  there is an unlabelled region  $R$ , then a disk swap of  $S_x$  gives the conclusion of Theorem 1. Hence in the remainder of this article, we show that there exists an unlabelled region.

Claim 2. No region can have both of the labels " $D_i$ " and " $E_{1-i}$ " ( $i=0$  or  $1$ ).

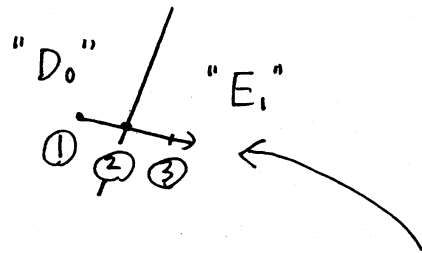
Proof. Suppose that there exists a region  $R$  with labels " $D_0$ " and " $E_1$ ". Hence there is a red compressing disk  $D(CS_x)$ , and a blue compressing disk  $E(CS_x)$ , where  $x \in R$  and  $[D]$  ( $[E]$  resp.) is a vertex of an element of  $\mathcal{C}_0$  ( $\mathcal{C}_1$  resp.).



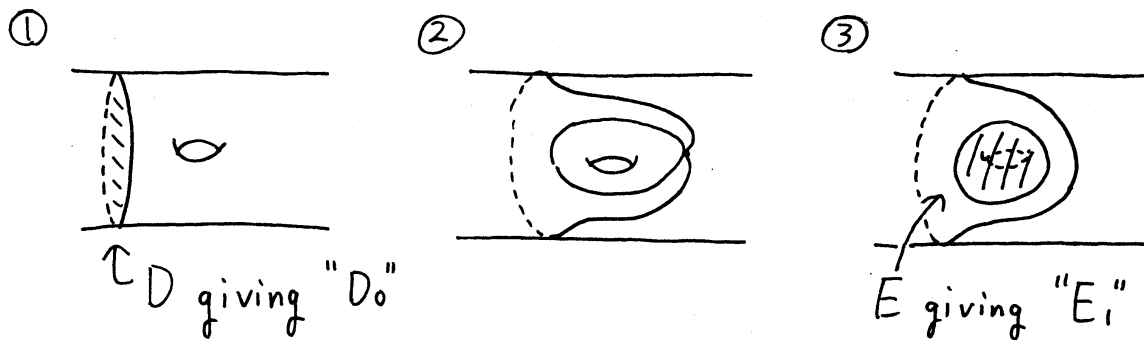
Since  $D \cap E \neq \emptyset$ , there is an edge  $[D] - [E]$ . However this contradicts the fact that  $[D]$  and  $[E]$  are contained in different components of  $\Gamma(F)$ . //

Claim 3. If a region is labelled by " $D_i$ ", then no adjacent region can be labelled by " $E_{1-i}$ ".

roof. Suppose this is not true. Hence



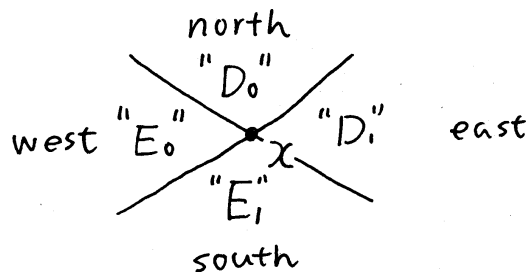
Consider the deformation of  $S$  along this path. Then  $S_x \cap F$  changes as follows.



By isotopy, we can make  $\partial D \cap \partial E \neq \emptyset$ . Hence there is an edge  $[D] - [E]$ , a contradiction. //

Claim 4 All four labels cannot occur around a vertex of  $\Sigma$ .

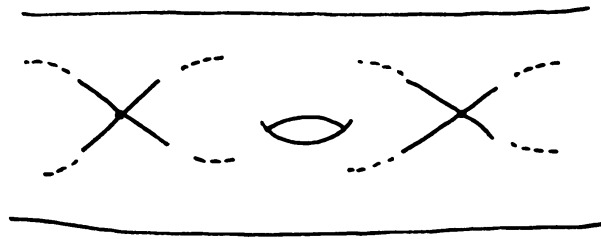
Proof. Suppose that this is not true. Call the regions north, east, south, and west (see figure below).



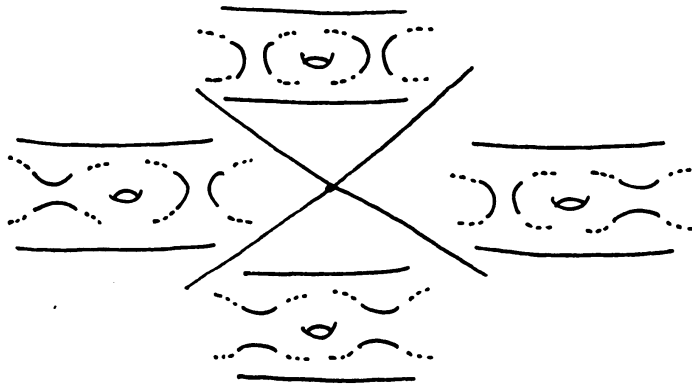
By Claims 2, 3 we see that the labelling must be as above

Then  $S_x \cap F$  contains two saddle tangencies (and we schematic

draw the intersection as follows).



Then in each region  $S_x \cap F$  will look as follows.



It is elementary to see that either

1. "north intersection" and "south intersection" can be made disjoint on  $F$  by an isotopy, or
2. "east intersection" and "west intersection" can be made disjoint on  $F$  by an isotopy.

In either case, we have a contradiction as in Claim 2. //

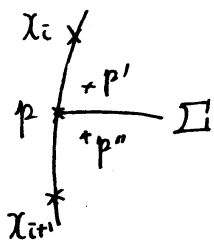
Claim 5. Let  $p \in \partial D^2$  be a point contained in a region, say  $R$ . Suppose that  $p$  lies between  $x_i$  and  $x_{i+1}$  on  $\partial D^2$ . Then  $R$  cannot have both of the labels " $E_0$ " and " $E_1$ ". (Corresponding statement holds for points between  $y_i$  and  $y_{i+1}$ .)

Proof. By the definition of  $\mathbb{F}$ , we may suppose without loss of generality that  $S_p \cap U_i = T_i^i(S) \cap U_i$  (hence,  $S_p \cap D_{\tilde{m}}^i = \emptyset \dots \textcircled{1}$ ). Suppose that  $R$  has both labels " $E_0$ " and " $E_1$ ", and  $E'(C S_p)$ ,  $E''(C S_p)$  be disks giving the labels " $E_0$ ", " $E_1$ " respectively. Since  $[E']$  and  $[E'']$  are contained in different components of  $T(F)$ , we may suppose  $[D_{\tilde{m}}^i]$  and  $[E']$  are contained in different components of  $T(F)$ . However the above  $\textcircled{1}$  shows that there is an edge  $[E'] - [D_{\tilde{m}}^i]$ , a contradiction. //

Claim 6. Let  $p$  be a point of  $\Sigma \cap \partial D^2$ . Suppose that  $R$  lies between  $x_i$  and  $x_{i+1}$  on  $\partial D^2$ . Let  $R_0$  and  $R_1$  be regions adjacent to  $p$ . Then  $\{\text{labels of } R_0\} \cup \{\text{labels of } R_1\}$  does not contain  $\{^{\wedge} E_0, ^{\wedge} E_1\}$ .

(Corresponding statement holds for points between  $y_i$  and  $y_{i+1}$ .)

Proof. Suppose that this is not the case. By Claim 5, we may suppose that  $R_0$  is labelled by " $E_0$ ", and  $R_1$  is labelled by " $E_1$ ", and let  $E'(C S_{p'})$ ,  $E''(C S_{p''})$  be disks giving the labels " $E_0$ ", " $E_1$ " respectively, where  $p' \in R_0$ ,  $p'' \in R_1$  are points close to  $p$ . As in the proof of Claim 5, we may



suppose that  $S_p \cap D_{\tilde{m}}^i = \emptyset$ . Hence we have

$$[E'] - [D_{\tilde{m}}^i] - [E'']$$

Hence  $[E']$  and  $[E'']$  are contained in the same component of  $T(F)$ , a contradiction. //

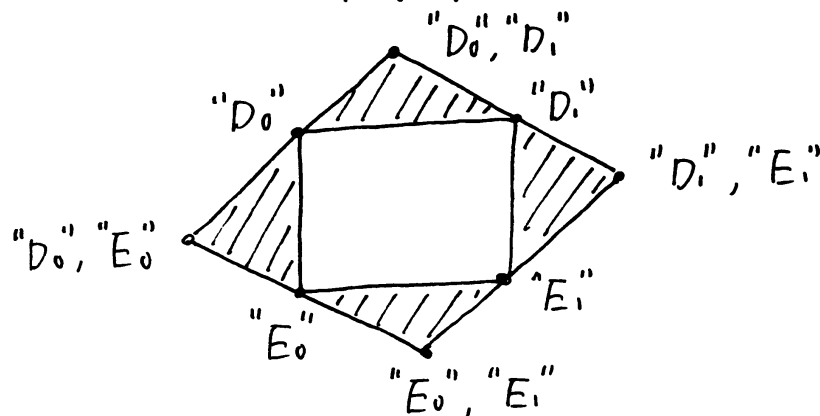
Claim 7. Let  $\theta_0 \in \partial D^2$  be a point between  $x_0$  and  $y_0$  such that  $S_{\theta_0} \cap U_0 = T_1^0(S) \cap U_0$ , and  $S_{\theta_0} \cap V_0 = \delta_1^0(S) \cap V_0$ , and let  $R_0$  be the region containing  $\theta_0$ . Then  $R_0$  cannot be labelled either by " $D_i$ " or " $E_i$ ".

(Corresponding statement holds for point  $\theta_1$  between  $x_m$  and  $y_m$ .)

Proof. Suppose that  $R_0$  is labelled by " $D_i$ " with corresponding disk  $D_i \subset S_{\theta_0}$ . Since  $S_{\theta_0} \cap E_0 \neq \emptyset$ , we see that there is an edge  $[D_i] - [E_0]$ . However this contradicts the fact that  $[D_i] \in \mathcal{C}_1$  and  $[E_0] \in \mathcal{C}_0$ . We obtain a contradiction in the same manner for label " $E_i$ ". //

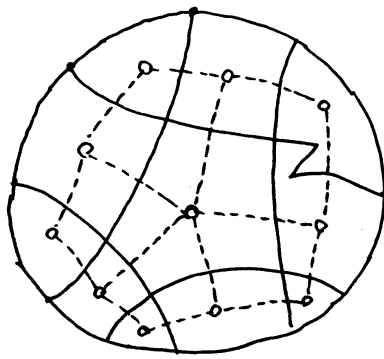
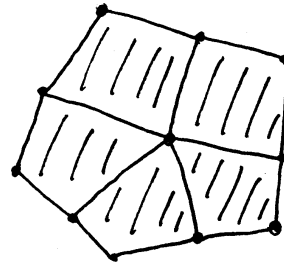
Stage 3. Finding an unlabelled region.

Suppose that each region is labelled. Let  $\Pi$  be the 2-complex with each vertex labelled as follows.



Let  $\Sigma'$  be the dual 2-complex of  $\Sigma$ .




 $\Sigma$ 

 $\Sigma'$ 

By Claim 2, we see that there is a map:  
 $\text{vertices}(\Sigma) \rightarrow \text{vertices}(\Pi),$

where the labellings are exactly those of the corresponding regions.

By Claim 3, we see that this map extends to:  
 $\text{edges}(\Sigma) \rightarrow \text{edges}(\Pi).$

By Claim 4, we see that this map extends to a simplicial map:  
 $\Sigma' \rightarrow \Pi.$

By Claims 5, 6, 7, we see that the restriction of the map to  $\partial\Sigma'$  is a degree 1 map. However this contradicts the fact that  $\Sigma'$  is simply connected.

## References

- [1] D. Bachman, *Critical Heegaard surfaces*, preprint
- [2] J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois* ( $\Gamma_4 = 0$ ), Lecture Notes in Math., 53(1968), Springer-Verlag, Berlin and New York.
- [3] K. Reidemeister, *Zur dreidimensionalen Topologie*, Abh. Math. Sem. Univ. Hamburg 9(1933), 189-194.
- [4] H. Rubinstein and M. Scharlemann, *Comparing Heegaard splittings of non-Haken 3-manifolds*, Topology **35** (1996), 1005 -1026.
- [5] J. Singer, *Three-dimensional manifolds and their Heegaard diagrams*, Trans. AMS 35(1933), 88-111.